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Comments on the Morita Equivalence

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It is known that noncommutative Yang-Mills theory with periodical boundary conditions on torus at the rational value of the noncommutativity parameter is Morita equivalent to the ordinary Yang-Mills theory with twisted boundary conditions on dual torus. We present simple derivation of this fact. We describe one-to-one correspondence between and gauge invariant observables in these two theories. In particular, we show that under Morita map Polyakov loops in the ordinary YM theory go to the open noncommutative Wilson loops discovered by Ishibashi, Iso, Kawai and Kutazawa.

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1 Introduction

Noncommutative geometry deals with functions on deformation of ordinary space, such that coordinates on it do not commute[2]:

$$[\hat{x}_\mu, \hat{x}_\nu] = 2\pi i \theta_{\mu\nu}, \quad \mu, \nu = 1, \dots d \tag{1.1}$$

The antisymmetrical tensor $\theta_{\mu\nu}$ is called noncommutativity parameter. Such deformed flat ($\theta_{\mu\nu} = \text{const}$) and compact space is called noncommutative (quantum) torus $\mathbf{T}^d_\theta$. In the last few years noncommutative geometry, and especially the noncommutative torus has been realized to play an important role in compactifications of M-theory [1] and in string theory (see [2] and references therein). It also turned to be very useful in compactification of instanton's moduli spaces [3]. The way to deal with the curved quantum spaces is provided by the Kontsevich's deformation quantization.

A very intriguing subject from noncommutative geometry is so-called *Morita equivalence* [4]. Roughly speaking, it states that certain bundles on different noncommutative tori are dual to each other. From the physical point of view it results in equivalence between certain noncommutative and ordinary gauge theories. In what follows we try to clarify this statement using a set of simple examples.

2 Notations

The algebra $\mathcal{A}_\theta$ of smooth functions on the noncommutative torus is defined using the Moyal star product:

$$f \star g\ (\hat{\mathbf{x}}) = e^{i\pi\theta_{\mu\nu}\frac{\partial}{\partial\xi_\mu}\frac{\partial}{\partial\eta_\nu}} f(\xi) g(\eta)\Big|_{\xi=\eta=\hat{\mathbf{x}}} \tag{2.1}$$

The main property of this product is associativity. In applications it is useful to decompose functions on noncommutative torus into the Fourier components[3]:

$$f(\hat{\mathbf{x}}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} f_{\mathbf{k}} e^{i\mathbf{k}\hat{\mathbf{x}}} \tag{2.2}$$

This corresponds to the Weil or symmetric ordering of coordinates. Exponents $\hat{U}_{\mathbf{k}} = e^{i\mathbf{k}\hat{\mathbf{x}}}$ may serve as a basis elements for the algebra $\mathcal{A}_\theta$.

A very intriguing thing happens when components of the $\theta-$tensor becomes rational. Let us first consider two-torus $\mathbf{T}^2$:

$$[\hat{x}_\mu, \hat{x}_\nu] = 2\pi i \theta \epsilon_{\mu\nu}, \quad \mu, \nu = 1, 2 \tag{2.3}$$

[2]In the sequel we use the same notation [,] both for ordinary and for star-commutator. To avoid confusions, we supply all noncommutative quantities with the hats.

[3]Without loss of generality we can consider a torus of size 2π.

with the rational noncommutativity parameter $\theta = \frac{M}{N}$, where M and N are relatively prime integers. Then

$$[\hat{U}_{\mathbf{n}}, \hat{U}_{\mathbf{n}'}] = 2i \sin\left(\pi M \frac{n_2 n_1' - n_1 n_2'}{N}\right) \hat{U}_{\mathbf{n}+\mathbf{n}'} = 2i \sin(\mathbf{n} \times \mathbf{n}')\, \hat{U}_{\mathbf{n}+\mathbf{n}'} \tag{2.4}$$

where by definition, $\mathbf{n} \times \mathbf{n}' \equiv -\pi \theta_{\mu\nu} n_\mu n'_\nu$. Note that elements $\hat{U}_{N\mathbf{k}}$ generate a center of the $\mathcal{A}_\theta$, that is for any $f(\hat{\mathbf{x}})$:

$$\left[e^{iN\mathbf{k}\hat{\mathbf{x}}}, f(\hat{\mathbf{x}})\right] = 0 \tag{2.5}$$

This means that one can treat exponents $\{\hat{U}_{\mathbf{k}},\ \mathbf{k} = 0|_{\mathrm{mod}\, N}\}$ in the decomposition (2.2) as if they are ordinary exponents defined on ordinary (commutative) space. Other N^2-1 exponents, obtained from the set $\{\hat{U}_{\mathbf{k}},\ \mathbf{k} \neq 0|_{\mathrm{mod}\, N}\}$ after factorization over commutative part, generates closed algebra under star-commutator. This algebra is isomorphic to the algebra of $SU(N)$, as we will see in a moment. Therefore, at the rational value of the noncommutativity parameter one can identify algebra of functions on the noncommutative torus with the algebra of matrix-valued functions on commutative torus.

We conclude this section by giving an explicit matrix representation for the noncommutative exponents algebra (see also [5]). Such a representation has been indeed well-known for many years [6, 7]. Let us introduce the following clock and shift generators

$$Q = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} \tag{2.6}$$

where $\omega = e^{2\pi i \theta}$. Matrices P and Q are unitary, traceless and satisfy:

$$P^N = Q^N = \mathbf{1}, \quad PQ = \omega\, QP \tag{2.7}$$

Moreover,

$$\mathrm{Tr}(P^n Q^m) = \begin{cases} N, & if \quad n = 0|_{\mathrm{mod}\, N} \quad and \quad m = 0|_{\mathrm{mod}\, N} \\ 0, & if \quad n \neq 0|_{\mathrm{mod}\, N} \quad or \quad m \neq 0|_{\mathrm{mod}\, N} \end{cases} \tag{2.8}$$

It is straightforward to check that the generators, defined as

$$J_{\mathbf{n}} = \omega^{\frac{n_1 n_2}{2}} Q^{n_1} P^{n_2}, \quad \mathbf{n} = (n_1, n_2) \tag{2.9}$$

satisfy commutation relations (2.4):

$$[J_{\mathbf{n}}, J_{\mathbf{n}'}] = 2i \sin\left(\mathbf{n} \times \mathbf{n}'\right) J_{\mathbf{n}+\mathbf{n}'} \tag{2.10}$$

This identity can be tautologically rewritten in the form of the Lie algebra commutation relations:

$$[J_{\mathbf{n}}, J_{\mathbf{m}}] = f_{\mathbf{nm}}^{\mathbf{k}} J_{\mathbf{k}}, \tag{2.11}$$

where the structure constants $f_{\mathbf{nm}}^{\mathbf{k}}$ are

$$f_{\mathbf{nm}}^{\mathbf{k}} = 2i\delta_{\mathbf{n}+\mathbf{m},\mathbf{k}} \sin(\mathbf{n} \times \mathbf{m}) \tag{2.12}$$

The set of unitary unimodular $N \times N$ matrices (2.9) suffices to span the algebra of $SU(N)$.

3 Morita Equivalence

3.1 Two-torus. $U(1)|_{\theta=\frac{M}{N}} \to U(N)$

To define Morita map we make an additional decomposition of the function (2.2) on the noncommutative two-torus:

$$\hat{f} = \sum_{k \in \mathbb{Z}^2} e^{iN\mathbf{k}\hat{\mathbf{x}}} \sum_{n_1,n_2=0}^{N-1} f_{\mathbf{k},\mathbf{n}} e^{in_1\hat{x}_1 + in_2\hat{x}_2} \tag{3.1}$$

Then we define corresponding $U(N)$-valued function on the ordinary two-torus as follows:

$$f = \sum_{k \in \mathbb{Z}^2} e^{iN\mathbf{k}\mathbf{x}} \sum_{n_1,n_2=0}^{N-1} f_{\mathbf{k},\mathbf{n}}\, e^{i\mathbf{n}\mathbf{x}}\ J_{\mathbf{n}} \tag{3.2}$$

Because of the relation

$$J_{\mathbf{n}} J_{\mathbf{n}'} = e^{\mathbf{i}\mathbf{n}\times\mathbf{n}'} J_{\mathbf{n}+\mathbf{n}'} \tag{3.3}$$

Morita map (3.1, 3.2) takes star-product to the matrix product. Obviously, $U(N)$-valued function of general type can not be represented in the form (3.2). It turns out that this particular form corresponds to the functions with nontrivial boundary conditions. Namely, under shifts these functions transforms as

$$f(x_1+2\pi\frac{M}{N},x_2)=\Omega_1\ f(x_1,x_2)\ \Omega_1^\dagger,\quad f(x_1,x_2+2\pi\frac{M}{N})=\Omega_2\ f(x_1,x_2)\ \Omega_2^\dagger \tag{3.4}$$

where

$$\Omega_1=(P)^M,\quad \Omega_2=(Q^\dagger)^M \tag{3.5}$$

This can be treated as a constant gauge transformation. The size $2\pi\frac{M}{N}$ of the dual torus can be fixed by the requirement for the Morita map to be single-valued[4]. To illustrate this, let us consider a torus of the size $2\pi\frac{M}{N}n$ (where $n\in\mathbf{N}$; there are no other possibilities if we want functions of the type (3.2) to be gauge transformed by the constant matrix when x is shifted by a period of the torus.) Obviously, in this case there are functions which cannot be represented in the form (3.2). Such functions do not conjugates when translated along the vectors $(2\pi\frac{M}{N},0)$ and $(0,2\pi\frac{M}{N})$.

Therefore, having a set of Fourier coefficients $f_{\mathbf{k},\mathbf{n}}$, we can construct both a function on the noncommutative torus of size l and a matrix-valued function with twisted boundary conditions (3.4) on the commutative torus of size $\frac{M}{N}l$ by the following rule:

$$\begin{cases} e^{i\mathbf{n}\hat{\mathbf{x}}}\ \leftrightarrow\ e^{i\mathbf{n}\mathbf{x}}\,J_{\mathbf{n}}, \quad n_1,n_2<N \\ \\ e^{iN\mathbf{k}\hat{\mathbf{x}}}\ \leftrightarrow\ e^{iN\mathbf{k}\mathbf{x}}\ \mathbf{1} \end{cases} \tag{3.6}$$

3.2 $\mathbf{T}^d$. $U(1)|_\theta\to U(N_1)\times\ldots\times U(N_r)$.

Generalization to the d-dimensional case goes by simple modifications in formulas from the previous subsection. It is always possible to rotate $\theta_{\mu\nu}$ into a canonical skew-diagonal form:

$$\theta_{\mu\nu}=\begin{pmatrix} 0 & \theta_1 & & & & \\ -\theta_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \theta_r & \\ & & & -\theta_r & 0 & \\ & & & & & \mathbf{0}_{d-2r} \end{pmatrix} \tag{3.7}$$

where r is the rank of $\theta_{\mu\nu}$. Thus, algebra of higher dimensional noncommutative torus becomes embedded into a d-fold tensor product of r noncommutative two-tori algebras and ordinary $(d-2r)$-torus commutative algebra. This immediately leads to other examples of Morita equivalence, when some of these noncommutative two-tori are mapped to the commutative ones using relations (3.6). If $\theta_i=\frac{M_i}{N_i}$, after Morita map we obtain an ordinary YM theory with the gauge group $U(N_1)\times\ldots\times U(N_r)$.

[4]I am indebted to K. Selivanov for this comment.

3.3 $\mathbf{T}^d$. $U(1)|_\theta \to U(N)$.

Algebra of noncommutative exponents can also be realized using a set of $SU(N)-$valued matrices $\Omega_\mu,\ \mu = 1, \ldots, d$ obeying the following relations:

$$\Omega_\mu \Omega_\nu = e^{2\pi i \theta_{\mu\nu}} \Omega_\nu \Omega_\mu \tag{3.8}$$

Explicit construction of such matrices can be found in [8]. Define generators $J_{\mathbf{n}}$ as follows:

$$J_{\mathbf{n}} = \exp\left(\sum_{\nu<\mu} \theta_{\nu\mu} n_\nu n_\mu\right)\ \Omega_1^{n_1} \ldots \Omega_d^{n_d} \tag{3.9}$$

Then

$$[J_{\mathbf{n}}, J_{\mathbf{m}}] = 2i \sin\left(\mathbf{n} \times \mathbf{m}\right) J_{\mathbf{n}+\mathbf{m}} \tag{3.10}$$

which coincides with the algebra of noncommutative exponents. Therefore, in this case Morita map takes the form:

$$\hat{f} = \sum_{k \in \mathbb{Z}^d} e^{iN\mathbf{k}\hat{\mathbf{x}}} \sum_{\mathbf{n}<N^{\otimes d}} f_{\mathbf{k},\mathbf{n}} e^{i\mathbf{n}\hat{\mathbf{x}}} \quad \leftrightarrow \quad f = \sum_{k \in \mathbb{Z}^d} e^{iN\mathbf{k}\mathbf{x}} \sum_{\mathbf{n}<N^{\otimes d}} f_{\mathbf{k},\mathbf{n}}\ e^{i\mathbf{n}\mathbf{x}}\ J_{\mathbf{n}} \tag{3.11}$$

4 Noncommutative YM vs Ordinary YM

Let us now turn to the physical applications of the Morita map. One can define noncommutative version of the Yang-Mills theory with the action

$$S_{YM} = \frac{1}{4\pi g_{YM}^2} \int d\mathbf{x}\ \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) \tag{4.1}$$

just by replacing in all formulas matrix product by the Moyal star-product and supplementing all quantities with the hats. Therefore, noncommutative $U(1)$ Yang-Mills action is

$$\hat{S} = \frac{1}{4\pi g_{NCYM}^2} \int d\hat{\mathbf{x}}\ \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} \tag{4.2}$$

where $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_\star$. For simplicity in this section we consider only two-torus. Generalization to the higher-dimensional case is straightforward.

Morita map takes NC $U(1)$ gauge fields to the $U(N)$ gauge fields with nontrivial boundary conditions. Generally, functions on torus can became gauge transformed when shifted by a period of the torus:

$$A_\lambda(\mathbf{x}+\mathbf{l}_\mu) = \Omega_\mu(\mathbf{x})\, A_\lambda(\mathbf{x})\, \Omega_\mu^{-1}(\mathbf{x}) + i\Omega_\mu(\mathbf{x})\, \partial_\lambda\, \Omega_\mu^{-1}(\mathbf{x}) \tag{4.3}$$

where $\Omega_\mu(\mathbf{x})$ are elements of the $U(N)$ group, known as twist matrices. They should satisfy a consistency conditions:

$$\Omega_\mu(\mathbf{x}+\mathbf{l}_\nu)\, \Omega_\nu(\mathbf{x}) = e^{2\pi i \frac{M}{N}\epsilon_{\mu\nu}}\, \Omega_\nu(\mathbf{x}+\mathbf{l}_\mu)\, \Omega_\mu(\mathbf{x}) \tag{4.4}$$

An integer M in this formula is so-called 't Hooft's flux. It is known only three types of possible boundary conditions (solutions of the eqs (4.4)):

1. *twist eaters*: $\Omega_\mu = \text{const}$
2. *abelian twists*
3. *nonabelian twists*

For more details see the recent review [9].

The map (3.6) corresponds exactly to the first case. It is not well understood how to realize Morita map corresponding to the other boundary conditions. Roughly speaking, when working in the Fourier basis (2.2), after shifts one can only multiply functions on numbers and cannot add something like $\Omega_\mu(\mathbf{x})\, \partial_\lambda\, \Omega_\mu^{-1}(\mathbf{x})$. To do this, one needs another basis for the functions on noncommutative torus (creation/annihilation operators, noncommutative theta-functions?).

Under Morita map, defined in the previous section, actions go to the actions, equations of motions go to the equations of motions, and solutions (e.g. instantons) also go to the solutions, even at the quantum level. These properties of the Morita map can be encoded in the following identity:

$$\int d\hat{\mathbf{x}}\ \hat{A}_\mu \star \hat{A}_\nu \star \ldots \star \hat{A}_\lambda = \frac{1}{N}\int d\mathbf{x}\ \operatorname{Tr}(A_\mu A_\nu \ldots A_\lambda) \tag{4.5}$$

which is straightforward to prove using the definition

$$\int d\hat{\mathbf{x}}\ e^{i\mathbf{k}\hat{\mathbf{x}}} = \delta_{\mathbf{k},\mathbf{0}} \tag{4.6}$$

and the property (2.8) of the clock and shift generators. In fact, one can insert arbitrary number of derivatives into the integrals in (4.5) and thus obtain equivalent gauge invariant quantities in noncommutative and ordinary gauge theories. Due to the identity (4.5) we can establish the following correspondence between correlators:

$$\int \mathcal{D}A^\mu_{\mathbf{k},\mathbf{n}}\ e^{\hat{S}[\theta=\frac{M}{N}]}\ \hat{\mathcal{O}}_1 \ldots \hat{\mathcal{O}}_l = \int \mathcal{D}A^\mu_{\mathbf{k},\mathbf{n}}\ e^{S_{YM}}\Big|_{fxd\ bndry\ conds,\ flux=M}\ \mathcal{O}_1 \ldots \mathcal{O}_l \tag{4.7}$$

where $g^2_{NCYM} = N g^2_{YM}$, and

$$\hat{\mathcal{O}} = \int d\hat{\mathbf{x}}\ (\hat{F}_{\mu\nu})^{\star n}, \quad \mathcal{O} = \frac{1}{N}\int d\mathbf{x}\ \operatorname{Tr}(F_{\mu\nu})^n \tag{4.8}$$

Other important gauge invariant quantities of the YM theory are the Wilson loops:

$$W[C] = \mathrm{Tr}\; P\exp\left(i\oint_C A_\mu(\mathbf{x})\,dx_\mu\right) \tag{4.9}$$

which corresponds to the closed path C. On torus there are paths from the different homotopy classes, which can be classified by winding numbers w_μ around the μ-th direction. The corresponding Wilson loops are called Polyakov loops. The simplest Polyakov loop corresponds to the straight line along the μ-th direction:

$$W_P[\mathbf{x},\mu] = \mathrm{Tr}\left[P\exp\left(i\int\limits_{\mathbf{x}}^{\mathbf{x}+\mathbf{l}_\mu} A_\mu(\mathbf{x})\,dx_\mu\right)\Omega_\mu e^{ix_\mu}\right] \tag{4.10}$$

where insertion of the twist matrix (3.5) is necessary to guarantee gauge invariance.

Wilson lines in noncommutative Yang-Mills theory were constructed by Ishibashi, Iso, Kawai and Kutazawa [10] (see also [11, 12]). This construction goes as follows. First, introduce an oriented curve C in auxiliary commutative two-dimensional space parametrized by the functions $\xi(\sigma)$ with $0 \leq \sigma \leq 1$. Fix the starting point $\xi_\mu(0) = 0$ end the endpoint $\xi_\mu(1) = v_\mu$. Then, assign to this curve a noncommutative analog of the parallel transport operator:

$$\begin{aligned} \mathcal{U}[\hat{\mathbf{x}},C] = 1 + \sum_{n=1}^{\infty} i^n \int\limits_0^1 d\sigma_1 \int\limits_{\sigma_1}^1 d\sigma_2\, \ldots \int\limits_{\sigma_{n-1}}^1 d\sigma_n\, \frac{d\xi_{\mu_1}(\sigma_1)}{d\sigma_1}\cdots\frac{d\xi_{\mu_n}(\sigma_n)}{d\sigma_n} \\ \times A_{\mu_1}(\hat{\mathbf{x}}+\xi(\sigma_{\mathbf{1}}))\star\ldots\star A_{\mu_n}(\hat{\mathbf{x}}+\xi(\sigma_{\mathbf{n}})) \end{aligned} \tag{4.11}$$

The series in (4.11) is noncommutative analog of the P-exponent. The star-gauge invariant quantity is then

$$\hat{\mathcal{O}}[C] = \int d\hat{\mathbf{x}}\;\mathcal{U}[\hat{\mathbf{x}},C]\star S[\hat{\mathbf{x}},C] \tag{4.12}$$

where $S[\hat{\mathbf{x}},C] = 1$ if the path C is closed and

$$S[\hat{\mathbf{x}},C] = e^{i(\theta^{-1})_{\mu\nu}v_\nu\hat{x}_\mu} \tag{4.13}$$

if the path is open. Gauge invariance requires that the coordinates of the endpoint must be equal to $v_\mu = 2\pi r_\mu \frac{M}{N}, \quad r_\mu = 0,\ldots,N-1$. In the simplest case, when C_μ is the straight line along the μ-th direction and $v_\mu = 2\pi\frac{M}{N}$, the function $S[\hat{\mathbf{x}},C_\mu]$ under Morita map (3.6) go to the twist function $\Omega_\mu e^{ix_\mu}$. Therefore, using identity (4.5) we obtain the following relation between the Polyakov loops in the ordinary YM theory and open noncommutative Wilson loops:

$$\frac{1}{N}\int d\mathbf{x}\;W_P[\mathbf{x},\mu] = \hat{\mathcal{O}}[C_\mu] \tag{4.14}$$

5 Conclusions

In this paper we have made some comments on the Morita equivalence between noncommutative and ordinary gauge theories. We present a simple prescription how to identify gauge fields and correlators of the gauge invariant observables in the $U(1)$ NC YM theory on torus at the rational value of the θ-parameter with those ones in the ordinary $U(N)$ or $U(N_1) \times \ldots \times U(N_r)$ YM theory with nontrivial boundary conditioxns on the dual torus. The size of the dual torus is determined by the requirement for the Morita map to be single-valued. We also show that under Morita map Polyakov loops in the ordinary YM theory go to the open noncommutative Wilson loops[5].

An open question is to generalize Morita equivalence to the case of the non-twist-eater's type boundary conditions. Another interesting direction is to link three different descriptions of the Morita equivalence: field theory approach using the Fourier components, string theory approach using T-duality and brane language [13, 14], and mathematical approach via the twisted bundles over the noncommutative torus [4, 15].

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[5]This fact firstly was mentioned in [12]

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